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THE BUCKLING OF PLATES AND BEAMS BY THE METHOD OF ZERO LAGRANGIAN MULTIPLIERS AND ZERO DIVISORS*

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Abstract—It is shown on a simple buckling problem, that the method of Lagrangian multipliers, which was assumed to be a simplification of the method of intermediate problems, actually yields incorrect numerical results.

1. INTRODUCTION

THE well-known Rayleigh–Ritz method gives *upper* bounds for the non-dimensional buckling load Λ of a clamped square plate compressed in all directions. In 1935 the present author [1] found extremely close *lower* bounds λ for the same quantity by a procedure which was later called, *The Method of Intermediate Problems.* In this way we obtained the inequalities $\lambda = 5{\cdot}30362 \leq \Lambda \leq 5{\cdot}31173$. This method has since been successfully applied not only to buckling and vibration problems but to quantum mechanics as well, see for instance the book of Gould [2] and the papers [3–7].

In order to use the method of intermediate problems we require a known problem which is in the present case the buckling of a supported plate. This problem gives rough lower bounds and is called the *base problem.* By adding some constraints we obtain intermediate problems, which can be explicitly solved using the base problem and which give improving lower bounds.

Later Trefftz [8, 9] reconsidered the problem and suggested an apparently simplified version—the so-called method of Lagrange multipliers—of our earlier approach. Other authors, (for instance, Budiansky and Hu [10]; see also [11–14]) have used and extended the method of Lagrangian multipliers which by now has found its way into a number of texts and handbooks on structural mechanics. Unfortunately, these authors fail to point out the care needed in applying the method to avoid possible errors in calculation of lower bounds. In the present paper, we show, through a simple example, the dangers that exist and how they can be circumvented by the method of intermediate problems.

We shall retain the terminology of [10], even though such multipliers occur in most variational problems and are not exclusively related to buckling.

2. ILLUSTRATIVE EXAMPLE OF THE BUCKLING OF A CLAMPED BEAM

We shall consider the Trefftz procedure in the simplest case of the buckling of a clamped beam in agreement with the statement [10, p.7] that a simple example should be used

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"*in order that the method of application of Lagrangian multipliers may be most clearly presented without the obscuring details of analysis of more complicated problems.*" Just as in the preliminary illustrative example of [10] the exact solution is known but in order to present our analysis we shall not use it.

Since we are using non-dimensional quantities, we consider a beam of length π, $0 \leq x \leq \pi$. We denote by $w(x)$ the deflection of the beam and put

$$M(w) = \int_0^\pi (w'')^2 \, dx$$

and

$$J(w) = \int_0^\pi (w')^2 \, dx.$$

The non-dimensional buckling load Λ is given by the minimum of $M(w)$ under the following side conditions:

$$J(w) = 1$$

$$w(0) = w(\pi) = 0 \tag{1}$$

$$w'(0) = w'(\pi) = 0. \tag{2}$$

(It is known that $\Lambda = 4$. However, what is not always realized, but is relevant for the following is that 4 is the second buckling load of the supported beam.)

The most general function satisfying the condition (1) is given by

$$w(x) = \sum_{\mu=1}^{\infty} a_\mu \sin \mu x.$$

Following [10], we replace the condition (2) by the conditions

$$\sum_{\mu=1}^{\infty} \mu a_\mu = 0, \qquad \sum_{\mu=1}^{\infty} (-1)^\mu \mu a_\mu = 0.$$

These lead to the two side conditions

$$\sum r a_r = 0, r = 1, 3, 5, \ldots \tag{3}$$

and

$$\sum s a_s = 0, s = 2, 4, 6, \ldots. \tag{4}$$

Taking the two conditions (3) and (4), we ought to obtain the exact value. However, following the procedure of Trefftz and others, we shall actually obtain a value not only greater than the exact value, but greater than the easily computed Rayleigh–Ritz upper bound 4·3, *a result which is incorrect.*

We denote by λ the allegedly unknown buckling load and introduce the Lagrangian multipliers $\pi\gamma_1$ and $\pi\gamma_2$. We consider the quantity

$$L(w) = M(w) - \lambda J(w) - \pi\gamma_1 \sum_{r=1,3,5,\ldots} r a_r - \pi\gamma_2 \sum_{s=2,4,6,\ldots} s a_s$$

or

$$L = \frac{\pi}{2}\sum_{\mu=1}^{\infty}\mu^4 a_\mu^2 - \frac{\lambda\pi}{2}\sum_{\mu=1}^{\infty}\mu^2 a_\mu^2 - \pi\gamma_1 \sum_{r=1,3,5,\dots} r a_r - \pi\gamma_2 \sum_{s=2,4,6,\dots} s a_s.$$

In order to find the unknowns $a_1, a_2, \dots$ and λ we set $\partial L/\partial a_\mu = 0$, $\mu = 1, 2, \dots$, and obtain the following sets of equations:

$$r^2(r^2-\lambda)a_r = \gamma_1 r; \qquad r = 1, 3, 5, \dots \tag{5}$$

$$s^2(s^2-\lambda)a_s = \gamma_2 s; \qquad s = 2, 4, 6, \dots. \tag{6}$$

Trefftz adds in his paper an incorrect remark excluding the possibility of all multipliers being zero, in the present case $\gamma_1 = \gamma_2 = 0$. His reasoning in our case would be that if $\gamma_1 = \gamma_2 = 0$, then by (5) and (6) all a_μ would be zero. However, he overlooks the fact that the value of λ could be such that one of the factors in his equations corresponding to our (5) and (6) is zero.

Let us prove in a correct way that γ_1 and γ_2 are not both zero. As we know already that $\lambda \leq 4{\cdot}3$, we must consider only the possibilities $\lambda = 1^2$ and $\lambda = 2^2$, which are the first and second buckling loads of a supported beam.

If $\lambda = 1$, then by (5) and (6) we have $a_2 = a_3 = a_4 = \dots = 0$ and a_1 is seemingly arbitrary since it satisfies the equation $0 . a_1 = 0$. However, from the condition (3) it follows that $a_1 = 0$, so that $w(x) \equiv 0$. Similarly, if $\lambda = 4$, we again get $w(x) \equiv 0$. Let us note in passing that Budiansky and Hu do not consider the vanishing of multipliers in their illustrative example.

Following the possibly erroneous procedure by *neglecting the possibility of zero divisors* [10, p. 12], *solving* (5) *and* (6) *and substituting in* (3) *and* (4) we obtain the equations

$$\begin{aligned} &\gamma_1 \sum \frac{1}{r^2-\lambda} = 0, \qquad r = 1, 3, 5 \dots \\ &\gamma_2 \sum \frac{1}{s^2-\lambda} = 0, \qquad s = 2, 4, 6, \dots \end{aligned} \tag{7}$$

with finite coefficients for γ_1 and γ_2.

Since the case $\gamma_1 = \gamma_2 = 0$ has been correctly excluded, the determinant $W(\lambda)$ of the pair of equations (7), which is the product of the two series, must be zero. As we know the closed forms of these series from the calculus, we have the equation $W(\lambda) = 0$ or

$$\left[\frac{\pi}{4\sqrt{\lambda}}\tan\frac{\pi\sqrt{\lambda}}{2}\right]\left[\frac{1}{2\lambda} - \frac{\pi}{4\sqrt{\lambda}}\cot\frac{\pi\sqrt{\lambda}}{2}\right] = 0 \tag{8}$$

which can be written as

$$\frac{\pi}{8\lambda\sqrt{\lambda}}\left[\tan\frac{\pi\sqrt{\lambda}}{2} - \frac{\pi\sqrt{\lambda}}{2}\right] = 0. \tag{9}$$

According to Trefftz, we have now to find the lowest positive root of the equation (8).

The first bracket in (8) has the roots $2^2, 4^2, 6^2, \dots$, but the second bracket is infinite for $\lambda = 2^2, 4^2, 6^2, \dots$ so that the value of the product is equal to $-\pi^2/(16)(2)^2$, $-\pi^2/(16)(4)^2, \dots$, as can be seen also from (9). Therefore the smallest root is given by the smallest zero of (9), and is definitely greater than 6·25. Let us emphasize here that the Trefftz procedure is not the Rayleigh–Ritz method. Therefore, the value $\lambda > 6{\cdot}25 > 4{\cdot}3$ shows that the tacit assumption of non-vanishing divisors leads to *erroneous results.*

We shall now give the correct solution by adapting the method of intermediate problems to the present case. As $\lambda \leq 4{\cdot}3$, there is no reason to exclude the possible values $\lambda = 1^2$ or $\lambda = 2^2$ in (5) and (6).

Assuming for the moment that $\lambda = 1$, we see from (5) that $0 \,.\, a_1 = \gamma_1$, from which it follows that $\gamma_1 = 0$ and that a_1 is seemingly arbitrary. It follows that $a_3 = a_5 = \ldots = 0$. Moreover, in order to satisfy the equation (3), a_1 must also be zero. As to the equation (6) none of the factors $(s^2 - \lambda)$ is zero for $\lambda = 1$ and therefore $\gamma_2 \neq 0$ since not all a_s are zero. Therefore, the divisors are not zero but the equation

$$\sum \frac{1}{s^2 - \lambda} = 0, \qquad s = 2, 4, 6, \ldots$$

is not satisfied for $\lambda = 1$.

Now consider the possibility $\lambda = 4$. Using the same reasoning as above we find this time that all $a_s = 0$, $s = 2, 4, 6, \ldots$. Furthermore, we must check to see whether or not $\lambda = 4$ is the smallest positive root of the first factor in (8), namely, $\pi/4\sqrt{\lambda}\tan \pi\sqrt{\lambda}/2$. This is indeed true, so that we have proved that λ is actually equal to 4.

3. CONCLUDING REMARKS

In view of the preceding discussion of the buckling of the beam we believe it to be superfluous to discuss separately and in detail the other cases considered by Trefftz and others. Neither shall we discuss here some vibration problems for clamped plates in which the determination of higher frequencies is of interest. In most cases we would encounter the same difficulties in the equation corresponding to $W(\lambda) = 0$. Also the case of vanishing Lagrangian multipliers cannot always be excluded. The method of intermediate problems provides simple and clear rules which cover every eventuality without having to go through a labyrinth of many separate cases, as we did above.

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Абстракт—Доказывается на лростом лримере вылучивания, что метод множителей Лагранжа, который рассматривается как улрощение метода промежуточных задач, приводит в действитепьности к неточным численным результатам.